

On discontinuity of information characteristics of quantum systems and channels

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Abstract

Quantitative analysis of discontinuity of basic characteristics of quantum states and channels is presented.

First we consider general estimates for discontinuity jump (loss) of the von Neumann entropy for a given converging sequence of states. It is shown, in particular, that for any sequence the loss of entropy is upper bounded by the loss of mean energy (with the coefficient characterizing Hamiltonian of a system).

Then we prove that discontinuity jumps of several correlation and entanglement measures in composite quantum systems are upper bounded by loss of one of the marginal entropies (with a corresponding coefficient).

We also analyse discontinuity of the output entropy of a quantum operation and of basic information characteristics of a quantum channel with respect to simultaneous variations of an input state and of a channel.

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1 Introduction

One of the main difficulties in study of infinite-dimensional quantum systems consists in discontinuity of basic characteristics of quantum states and channels (such as von Neumann entropy, conditional entropy, quantum mutual information, entanglement measures, etc.). This shows necessity to find conditions for local continuity of such characteristic. The first results in this direction seems to be Simon's convergence theorems for the von Neumann entropy [1, the Appendix]. Since then many different continuity conditions for the entropy and other basic information quantities have been found (see [2, 3, 4, 5] and the references therein).

In this paper we present quantitative analysis of discontinuity of several important characteristics of quantum states and channels starting with the von Neumann entropy.

In Section 3 we consider general estimates for discontinuity jumps of the von Neumann entropy and an expression for these jumps based on the approximating technique (Propositions 1,3 and their corollaries). We also consider relations between discontinuity jumps of the entropy and majorization (Proposition 2). Then we focus attention on estimating discontinuity of the entropy on the set of states with bounded mean energy, i.e. states ρ satisfying the inequality

$$\text{Tr} H \rho \leq E \tag{1}$$

where H is the Hamiltonian of a system. It is well known that the entropy is continuous on this set if (and only if) $\text{Tr} e^{-\lambda H}$ is finite for all $\lambda > 0$ [3]. Explicit continuity bounds for the entropy on this set were recently obtained

by Winter [4]. We analyse discontinuity jumps (losses) of the entropy on the set determined by inequality (1) in the case of logarithmic growth of the eigenvalues of H , i.e. when

$$\mathrm{Tr} e^{-\lambda H} < +\infty \text{ for some } \lambda > 0. \quad (2)$$

It is shown that for any converging sequence of states the loss of entropy is upper bounded by the loss of mean energy with the coefficient $g(H)$ – the infimum of all λ in (2) (Proposition 4).

In Section 4 we show that discontinuity jumps of several measures of classical and quantum correlations in composite quantum systems are upper bounded by discontinuity jump of one of the marginal entropies (with a corresponding coefficient). The main conclusion obtained by joining these results and the observation from Section 3 can be briefly formulated as follows: *if Hamiltonians of quantum subsystems satisfy condition (2) then discontinuity of many characteristics of a composite quantum state is related to the loss of mean energy in one of the subsystems.*¹

In Section 5 we analyse discontinuity of the output entropy of a quantum operation and of the basic information characteristics of a quantum channel: the constrained Holevo capacity, the quantum mutual information and the coherent information. We obtain estimates for discontinuity jumps of these characteristics with respect to simultaneous variations of an input state and of a channel expressed via loss of the input (output) entropy.

2 Preliminaries

Let \mathcal{H} be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ – Banach spaces of all bounded operators and of all trace-class operators in \mathcal{H} , $\mathfrak{T}_+(\mathcal{H})$ – the cone of positive operators in $\mathfrak{T}(\mathcal{H})$, $\mathfrak{S}(\mathcal{H})$ – the set of quantum states (operators in $\mathfrak{T}_+(\mathcal{H})$ with unit trace) [2, 8].

Denote by $I_{\mathcal{H}}$ the identity operator in a Hilbert space \mathcal{H} and by $\mathrm{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

A *quantum operation* Φ from a system A to a system B is a completely positive trace non-increasing linear map $\mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$, where \mathcal{H}_A and \mathcal{H}_B

¹There exist correlation measures whose discontinuity *is not related* to discontinuity of the entropy (and hence to the loss of mean energy) [6, 7].

are Hilbert spaces associated with the systems A and B . A trace preserving quantum operation is called *quantum channel* [2, 8].

For any quantum channel $\Phi : A \rightarrow B$ Stinespring's theorem implies the existence of a Hilbert space \mathcal{H}_E and of an isometry $V : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_E V \rho V^*, \quad \rho \in \mathfrak{T}(\mathcal{H}_A). \quad (3)$$

The minimal dimension of \mathcal{H}_E is called *Choi rank* of Φ . The quantum channel

$$\mathfrak{T}(\mathcal{H}_A) \ni \rho \mapsto \widehat{\Phi}(\rho) = \text{Tr}_B V \rho V^* \in \mathfrak{T}(\mathcal{H}_E) \quad (4)$$

is called *complementary* to the channel Φ [2, Ch.6].

The *quantum relative entropy* for two operators ρ and σ in $\mathfrak{T}_+(\mathcal{H})$ is defined as follows (cf.[9])

$$H(\rho \parallel \sigma) = \sum_{i=1}^{+\infty} \langle i | \rho \log \rho - \rho \log \sigma + \sigma - \rho | i \rangle, \quad (5)$$

where $\{|i\rangle\}_{i=1}^{+\infty}$ is the orthonormal basis of eigenvectors of the operator ρ and it is assumed that $H(\rho \parallel \sigma) = +\infty$ if $\text{supp} \rho$ is not contained in $\text{supp} \sigma$. This definition implies $H(\lambda \rho \parallel \lambda \sigma) = \lambda H(\rho \parallel \sigma)$ for $\lambda \geq 0$.

We will use the following result of the purification theory [2].

Lemma 1. *Let \mathcal{H} and \mathcal{K} be Hilbert spaces such that $\dim \mathcal{H} = \dim \mathcal{K}$. For an arbitrary pure state ω_0 in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ and an arbitrary sequence $\{\rho_k\}$ of states in $\mathfrak{S}(\mathcal{H})$ converging to the state $\rho_0 = \text{Tr}_{\mathcal{K}} \omega_0$ there exists a sequence $\{\omega_k\}$ of pure states in $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ converging to the state ω_0 such that $\rho_k = \text{Tr}_{\mathcal{K}} \omega_k$ for all k .*

We will repeatedly use the following simple lemmas in which X is an arbitrary metric space.

Lemma 2. *Let f , g and h be functions on X such that $f + g = h$ and $\{x_n\}$ a sequence converging to x_0 such that $f(x_0)$, $g(x_0)$ and $h(x_0)$ are finite. If the function g is lower semicontinuous then*

$$\limsup_{n \rightarrow \infty} f(x_n) - f(x_0) \leq \limsup_{n \rightarrow \infty} h(x_n) - h(x_0)$$

If the function h is lower semicontinuous then

$$f(x_0) - \liminf_{n \rightarrow \infty} f(x_n) \leq \limsup_{n \rightarrow \infty} g(x_n) - g(x_0).$$

Proof. Since $f(x_n) + g(x_n) = h(x_n)$ for all n , we have

$$\limsup_{n \rightarrow \infty} f(x_n) + \liminf_{n \rightarrow \infty} g(x_n) \leq \limsup_{n \rightarrow \infty} h(x_n).$$

By substituting the equality $f(x_0) + g(x_0) = h(x_0)$ from this inequality and by using the lower semicontinuity of g we obtain the first assertion of the lemma. The second assertion follows from the first one with $f' = -f, h' = g, g' = h$. \square

Lemma 3. *Let $\{f_k\}$ and $\{g_k\}$ be nondecreasing sequences of continuous functions on X pointwise converging respectively to functions f and g . If $f(x) - f_k(x) \leq g(x) - g_k(x)$ for all $x \in X$ then*

$$\limsup_{n \rightarrow \infty} f(x_n) - f(x_0) \leq \limsup_{n \rightarrow \infty} g(x_n) - g(x_0)$$

for any sequence $\{x_n\}$ converging to a state x_0 such that $g(x_0) < +\infty$.

Proof. It suffices to note that continuity of the functions f_k and g_k imply

$$\limsup_{n \rightarrow \infty} h(x_n) - h(x_0) = \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} (h - h_k)(x_n), \quad h = f, g,$$

provided that $g(x_0)$ (and hence $f(x_0)$) are finite. \square

Lemma 4. *Let $\{f_k\}$ be a non-increasing sequence of functions on X pointwise converging to a function f and $\{x_n\}$ a sequence converging to x_0 such that $f(x_0) < +\infty$. If*

$$\limsup_{n \rightarrow \infty} f_k(x_n) - f_k(x_0) \leq C \text{ for all } k \text{ then } \limsup_{n \rightarrow \infty} f(x_n) - f(x_0) \leq C.$$

3 On discontinuity of the von Neumann entropy

3.1 General estimates

The *von Neumann entropy* $H(\rho) = \text{Tr} \eta(\rho)$, where $\eta(x) = -x \log x$, is a basic characteristic of a state $\rho \in \mathfrak{S}(\mathcal{H})$. It has the homogeneous extension to the

cone $\mathfrak{T}_+(\mathcal{H})$ (cf.[9])²

$$H(\rho) = [\text{Tr}\rho]H\left(\frac{\rho}{\text{Tr}\rho}\right) = \text{Tr}\eta(\rho) - \eta(\text{Tr}\rho), \quad \rho \in \mathfrak{T}_+(\mathcal{H}). \quad (6)$$

This extension naturally arises in applications, for example, in analysis of the output entropy of quantum trace-non-preserving operation (see Sect.5.1). Nonnegativity, concavity and lower semicontinuity of the von Neumann entropy on the cone $\mathfrak{T}_+(\mathcal{H})$ follow from the corresponding properties of this function on the set $\mathfrak{S}(\mathcal{H})$ [3, 9].

By the lower semicontinuity of the entropy $H(\rho)$ its discontinuity jumps for a given sequence $\{\rho_n\} \in \mathfrak{T}_+(\mathcal{H})$ converging to an operator ρ_0 can be characterised by the nonnegative value

$$\text{dj}\{H(\rho_n)\} \doteq \limsup_{n \rightarrow +\infty} H(\rho_n) - H(\rho_0),$$

where it is assumed that $\text{dj}\{H(\rho_n)\} = +\infty$ if $H(\rho_0) = +\infty$. This value can be called *the entropy loss* corresponding to the sequence $\{\rho_n\}$.³

We begin with the following simple but useful observation.

Proposition 1. *Let $\{\rho_n\} \in \mathfrak{T}_+(\mathcal{H})$ be a sequence converging to an operator ρ_0 . Then*

$$\text{dj}\{H(\rho_n)\} \leq \limsup_{n \rightarrow \infty} \text{Tr}\rho_n(-\log \sigma_n) - \text{Tr}\rho_0(-\log \sigma_0) \quad (7)$$

for any sequence $\{\sigma_n\} \in \mathfrak{T}_+(\mathcal{H})$ converging to an operator σ_0 , where it is assumed that the right hand side is equal to $+\infty$ if $\text{Tr}\rho_0(-\log \sigma_0) = +\infty$.

Note that " = " trivially holds in (7) if $\sigma_n = \rho_n$ for all n .

Proof. It follows from (5) and (6) that

$$H(\rho_n) + H(\rho_n \| \sigma_n) + f(\rho_n) - \text{Tr}\sigma_n = \text{Tr}\rho_n(-\log \sigma_n) \quad (8)$$

for all $n \geq 0$, where $f(\rho_n) = \eta(\text{Tr}\rho_n) + \text{Tr}\rho_n$. Hence

$$\limsup_{n \rightarrow \infty} H(\rho_n) + \liminf_{n \rightarrow \infty} H(\rho_n \| \sigma_n) + \lim_{n \rightarrow \infty} [f(\rho_n) - \text{Tr}\sigma_n] \leq \limsup_{n \rightarrow \infty} \text{Tr}\rho_n(-\log \sigma_n).$$

By subscribing equality (8) with $n = 0$ from this inequality and by using the lower semicontinuity of the relative entropy we obtain (7). \square

²Here and in what follows \log denotes the natural logarithm.

³The term "entropy loss" is used in literature in different senses [10, 11].

We will use Proposition 1 below. Now consider two simple corollaries.

Let $\{|k\rangle\}$ be an orthonormal basis in \mathcal{H} . For any state $\rho \in \mathfrak{S}(\mathcal{H})$ we may consider the probability distribution $\pi(\rho) = \{\langle k|\rho|k\rangle\}$. It is well known that $H(\rho) \leq S(\pi(\rho))$, where S is the Shannon entropy – a lower semicontinuous function on the set of all countable probability distributions equipped with the ℓ_1 metric. Proposition 1 shows that a similar relation hold for jumps of the entropy corresponding to converging sequences $\{\rho_n\}$ and $\{\pi(\rho_n)\}$.⁴

Corollary 1. *For any sequence $\{\rho_n\} \in \mathfrak{S}(\mathcal{H})$ converging to a state ρ_0 we have*

$$\text{dj}\{H(\rho_n)\} \leq \text{dj}\{S(\pi(\rho_n))\} \doteq \limsup_{n \rightarrow \infty} S(\pi(\rho_n)) - S(\pi(\rho_0)), \quad (9)$$

where it is assumed that $\text{dj}\{S(\pi(\rho_n))\}$ is equal to $+\infty$ if $S(\pi(\rho_0)) = +\infty$.

Note that “=” holds in (9) if the sequence $\{\rho_n\}$ consists of states diagonalisable in the basis $\{|k\rangle\}$.

Proof. It suffices to take $\sigma_n = \sum_k \langle k|\rho_n|k\rangle |k\rangle\langle k|$ for all $n \geq 0$, and to apply Proposition 1. \square

By subadditivity of the von Neumann entropy $H(\omega_{AB}) \leq H(\omega_A) + H(\omega_B)$ for any bipartite state ω_{AB} , where $\omega_A \doteq \text{Tr}_B \omega_{AB}$ and $\omega_B \doteq \text{Tr}_A \omega_{AB}$ are marginal states. Similar relation holds for jumps of the entropy.

Corollary 2. *Let $\{\omega_{AB}^n\}$ be a sequence of bipartite states converging to a state ω_{AB}^0 . Then*

$$\text{dj}\{H(\omega_{AB}^n)\} \leq \text{dj}\{H(\omega_A^n)\} + \text{dj}\{H(\omega_B^n)\}.$$

Proof. It suffices to take $\sigma_n = \omega_A^n \otimes \omega_B^n$ for all $n \geq 0$ and to apply Proposition 1. \square

The triangle inequalities $H(\omega_X) \leq H(\omega_{AB}) + H(\omega_Y)$, $XY = AB, BA$ and the implication $H(\omega_{AB}) = 0 \Rightarrow H(\omega_A) = H(\omega_B)$ have the following dj-versions

$$\text{dj}\{H(\omega_X^n)\} \leq \text{dj}\{H(\omega_{AB}^n)\} + 2\text{dj}\{H(\omega_Y^n)\}, \quad XY = AB, BA,$$

where the factor 2 can be removed if $\{H(\omega_Y^n)\}$ is a converging sequence, and

$$\text{dj}\{H(\omega_{AB}^n)\} = 0 \Rightarrow \text{dj}\{H(\omega_A^n)\} = \text{dj}\{H(\omega_B^n)\},$$

⁴It is easy to see that the map $\rho \mapsto \pi(\rho)$ is continuous.

valid for any sequence $\{\omega_{AB}^n\}$ converging to a state ω_{AB}^0 . These relations directly follow from Theorem 2 and Corollary 11 in Section 5 (where $\Phi_n = \Phi$ is a partial trace). The last implication means that *continuity of the bipartite entropy implies coincidence of the marginal entropy losses*.

The inequalities $H(\omega_X) \leq H(\omega_{AB})$, $X = A, B$, for a separable state ω_{AB} also have dj-versions (see Corollary 4B in the next subsection 3.2).

3.2 The entropy loss and majorization

Important role in quantum information theory is played by the special partial order between quantum states called *majorization* (see [12, 13, 14] and the references therein). We say that a state ρ majorizes a state σ and write $\rho \succ \sigma$ if

$$\sum_{k=1}^n \lambda_k \geq \sum_{k=1}^n \mu_k \quad \text{for all } n \in \mathbb{N}, \quad (10)$$

where $\{\lambda_k\}$ and $\{\mu_k\}$ are sequences of eigenvalues of ρ and σ taken in decreasing order. Denote these sequences respectively by ρ^\downarrow and σ^\downarrow .

It is well known that $\rho \succ \sigma$ implies $H(\rho) \leq H(\sigma)$ [12, 13]. To prove the analogous implication for jumps of the entropy we will use the following

Proposition 2. *Let D be the classical relative entropy (the Kullback-Leibler distance). If $\rho \succ \sigma$ then*

$$H(\sigma) = H(\rho) + D(\rho^\downarrow \| \sigma^\downarrow) + f(\rho, \sigma), \quad (11)$$

where $f(\rho, \sigma)$ is a nonnegative lower semicontinuous function on the closed subset⁵ $\mathfrak{S}_\succ \doteq \{(\rho, \sigma) \mid \rho \succ \sigma\}$ of $[\mathfrak{S}(\mathcal{H})]^{\times 2}$ well defined for states ρ and σ with finite entropy by the expression

$$f(\rho, \sigma) = \text{Tr}(\sigma^\downarrow - \rho^\downarrow)(-\log \sigma^\downarrow) = \sum_{k=1}^{+\infty} (\mu_k - \lambda_k)(-\log \mu_k).$$

Remark 1. Proposition 2 and Pinsker's inequality show that

$$\rho \succ \sigma \quad \Rightarrow \quad H(\sigma) - H(\rho) \geq D(\rho^\downarrow \| \sigma^\downarrow) \geq \frac{1}{2} \|\sigma^\downarrow - \rho^\downarrow\|_1^2.$$

⁵It is assumed that the set \mathfrak{S}_\succ is equipped with the Cartesian product topology.

This gives a simple proof of the strict monotonicity of the entropy with respect to majorization (cf. [13]).

By Proposition 2 for any sequences $\{\rho_n\}$ and $\{\sigma_n\}$ converging respectively to states ρ_0 and σ_0 such that $\rho_n \succ \sigma_n$ for all n we have

$$\liminf_{n \rightarrow \infty} f(\rho_n, \sigma_n) \geq f(\rho_0, \sigma_0).$$

This and the lower semicontinuity of the function $(\rho, \sigma) \mapsto D(\rho^\downarrow \parallel \sigma^\downarrow)$ make possible to derive from (11) the following observation.

Corollary 3. *Let $\{\rho_n\}$ and $\{\sigma_n\}$ be sequences of states converging respectively to states ρ_0 and σ_0 such that $\rho_n \succ \sigma_n$ for all n . Then*

$$\text{dj}\{H(\rho_n)\} \leq \text{dj}\{H(\sigma_n)\} - \Delta_1 - \Delta_2 \leq \text{dj}\{H(\sigma_n)\},$$

where

$$\Delta_1 = \liminf_{n \rightarrow \infty} D(\rho_n^\downarrow \parallel \sigma_n^\downarrow) - D(\rho_0^\downarrow \parallel \sigma_0^\downarrow) \geq 0, \quad \Delta_2 = \liminf_{n \rightarrow \infty} f(\rho_n, \sigma_n) - f(\rho_0, \sigma_0) \geq 0.$$

Proof of Proposition 2. If $H(\sigma) < +\infty$ then

$$H(\sigma) = H(\rho) + D(\rho^\downarrow \parallel \sigma^\downarrow) + \sum_{k=1}^{+\infty} (\mu_k - \lambda_k)(-\log \mu_k),$$

where the last series is nonnegative by the below Lemma 5. Lemma 5 also shows that this series is a limit of the nondecreasing sequence of nonnegative numbers

$$f_n(\rho, \sigma) = \sum_{k=1}^{+\infty} (\mu_k - \lambda_k) h_k^n, \quad h_k^n = \min\{n, -\log \mu_k\}.$$

The function $f_n(\rho, \sigma)$ is continuous on \mathfrak{S}_\succ for each n by Mirsky's inequality $\|\rho^\downarrow - \sigma^\downarrow\|_1 \leq \|\rho - \sigma\|_1$ [15]. So, the function $f(\rho, \sigma) \doteq \sup_n f_n(\rho, \sigma)$ possesses all the properties stated in the proposition. It suffices only to verify that if $H(\sigma) = +\infty$ but $H(\rho) < +\infty$ and $D(\rho^\downarrow \parallel \sigma^\downarrow) < +\infty$ then $f(\rho, \sigma) = +\infty$ \square .

Lemma 5. *Let $\{\lambda_k\}$ and $\{\mu_k\}$ be probability distributions such that $\{\lambda_k\} \succ \{\mu_k\}$. Then $\sum_{k=1}^{+\infty} \lambda_k h_k \leq \sum_{k=1}^{+\infty} \mu_k h_k$ for any nondecreasing sequence $\{h_k\}$ of nonnegative numbers.*

Proof. It suffices to note that $\sum_{k=1}^{+\infty} \nu_k h_k = \sum_{k=1}^{+\infty} d_k S_k^\nu + h_1$, $\nu = \lambda, \mu$, where $d_k = h_{k+1} - h_k \geq 0$ and $S_n^\nu = \sum_{k>n} \nu_k$, and to use (10). \square

Let $\mathfrak{S}_s(\mathcal{H}_{AB})$ be the set of all separable states in $\mathfrak{S}(\mathcal{H}_{AB})$ (defined as the convex closure of all product states in $\mathfrak{S}(\mathcal{H}_{AB})$). Theorem 11.0.1 in [12] states that

$$\omega_A \succ \omega_{AB} \quad \text{and} \quad \omega_B \succ \omega_{AB} \quad (12)$$

for any state $\omega_{AB} \in \mathfrak{S}_s(\mathcal{H}_{AB})$ provided \mathcal{H}_A and \mathcal{H}_B are finite-dimensional spaces. To generalize this theorem to the case $\dim \mathcal{H}_A = \dim \mathcal{H}_B = +\infty$ it suffices to approximate a separable state ω_{AB} by any sequence $\{\omega_{AB}^n\}$ of separable states with finite rank marginal states ω_A^n and ω_B^n and to note that $\omega_X^n \succ \omega_{AB}^n$ for all n implies $\omega_X \succ \omega_{AB}$, $X = A, B$.⁶

Relation (12), Proposition 2 and Corollary 3 imply the following

Corollary 4. *Let $\mathfrak{S}_s^f(\mathcal{H}_{AB}) = \{\omega_{AB} \in \mathfrak{S}_s(\mathcal{H}_{AB}) \mid H(\omega_{AB}) < +\infty\}$.*

A) *The functions $\omega_{AB} \mapsto H(\omega_{AB}) - H(\omega_X)$, $X = A, B$, are nonnegative and lower semicontinuous on the set $\mathfrak{S}_s^f(\mathcal{H}_{AB})$.*

B) *If $\{\omega_{AB}^n\}$ is a sequence of separable states converging to a state ω_{AB}^0 then*

$$\text{dj}\{H(\omega_X^n)\} \leq \text{dj}\{H(\omega_{AB}^n)\}, \quad X = A, B.$$

Corollaries 2 and 4B show that

$$\max\{\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_B^n)\}\} \leq \text{dj}\{H(\omega_{AB}^n)\} \leq \text{dj}\{H(\omega_A^n)\} + \text{dj}\{H(\omega_B^n)\}$$

for any sequence $\{\omega_{AB}^n\}$ of separable states converging to a state ω_{AB}^0 .

Corollary 4A makes possible to prove lower semicontinuity of the coherent information and of the entropy gain for all infinite-dimensional quantum channels complementary to entanglement-breaking channels. Following [16] we will call such channels *pseudo-diagonal*.

Corollary 5. *If $\Phi : A \rightarrow B$ is a pseudo-diagonal quantum channel then the coherent information $I_c(\Phi, \rho) \doteq H(\Phi(\rho)) - H(\widehat{\Phi}(\rho))$ and the entropy gain $EG(\Phi, \rho) \doteq H(\Phi(\rho)) - H(\rho)$ are nonnegative lower semicontinuous functions on the set $\{\rho \in \mathfrak{S}(\mathcal{H}_A) \mid H(\Phi(\rho)) < +\infty\}$.*

Proof. If ρ_{AR} is any purification of an input state ρ_A then

$$I_c(\Phi, \rho) \doteq H\left(\widehat{\Phi} \otimes \text{Id}_R(\rho_{AR})\right) - H(\widehat{\Phi}(\rho_A))$$

⁶This approximation is necessary because of the existence of countably nondecomposable separable states in infinite-dimensional bipartite system.

and

$$EG(\Phi, \rho) \doteq H\left(\widehat{\Phi} \otimes \text{Id}_R(\rho_{AR})\right) - H(\rho_A).$$

Since $\widehat{\Phi}$ is an entanglement-breaking channel, $\widehat{\Phi} \otimes \text{Id}_R(\rho_{AR})$ is a separable state in $\mathfrak{S}(\mathcal{H}_{ER})$. So, the assertions of the corollary follow from Corollary 4A and Lemma 1. \square

3.3 Use of the approximating technique

In [17] it is shown that the function $\mathfrak{T}_+(\mathcal{H}) \ni \rho \mapsto H(\rho)$ is a pointwise limit of the nondecreasing sequence of concave continuous functions

$$H_k(\rho) \doteq \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H(\rho_i), \quad \rho \in \mathfrak{T}_+(\mathcal{H}), \quad (13)$$

where $\mathcal{P}_k(\rho)$ is the sets of all countable ensembles of positive trace class operators of rank $\leq k$ with the average state ρ (if ρ is a state then the supremum in (13) can be taken over all countable ensembles of *states* of rank $\leq k$ with the average state ρ).

The function H_k may be called *k-approximator* of the von Neumann entropy. For any $\rho \in \mathfrak{T}_+(\mathcal{H})$ the difference $\Delta_k^H(\rho) = H(\rho) - H_k(\rho)$ between the von Neumann entropy and its *k-approximator* can be expressed as follows

$$\Delta_k^H(\rho) = \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H(\rho_i \| \rho), \quad (14)$$

where $H(\cdot \| \cdot)$ is the extended quantum relative entropy defined by (5).

The sequence $\{H_k\}$ is used in [17] for analysis of continuity of the von Neumann entropy. It can be also used for estimating discontinuity jumps of the entropy. Since the sequence $\{H_k\}$ pointwise converges to the function H and consists of continuous functions, expression (14) implies the following

Proposition 3. *Let $\{\rho_n\} \subset \mathfrak{T}_+(\mathcal{H})$ be a sequence converging to an operator ρ_0 with finite $H(\rho_0)$. Then $\text{dj}\{H(\rho_n)\} = \limsup_{n \rightarrow \infty} \Delta_k^H(\rho_n) - \Delta_k^H(\rho_0)$ for any k and hence*

$$\text{dj}\{H(\rho_n)\} = \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \Delta_k^H(\rho_n) = \inf_k \limsup_{n \rightarrow \infty} \Delta_k^H(\rho_n).$$

Applicability of Proposition 3 is based on special properties of the function Δ_k^H presented in Lemma 8 in [17]. These properties are derived by using

representation (14) from the well known analytical properties of the quantum relative entropy. For example, the joint convexity of the relative entropy implies

$$\Delta_{k+l}^H(\rho + \sigma) \leq \Delta_k^H(\rho) + \Delta_l^H(\sigma), \quad \rho, \sigma \in \mathfrak{T}_+(\mathcal{H}), \quad (15)$$

while the monotonicity of the relative entropy shows that

$$\rho \leq \sigma \Rightarrow \Delta_k^H(\rho) \leq \Delta_k^H(\sigma), \quad \rho, \sigma \in \mathfrak{T}_+(\mathcal{H}), \quad (16)$$

where " \leq " in the left side denotes the operator order, and that

$$\Delta_{mk}^H(\Phi(\rho)) \leq \Delta_k^H(\rho) \quad (17)$$

for any $\rho \in \mathfrak{T}_+(\mathcal{H})$ and any quantum operation Φ with Choi rank $\leq m$.

By Proposition 3 properties (15) and (16) imply the following

Corollary 6. *Let $\{\rho_n\}$ and $\{\sigma_n\}$ be sequences of operators in $\mathfrak{T}_+(\mathcal{H})$ converging respectively to operators ρ_0 and σ_0 . Then*

$$\max \{ \text{dj}\{H(\rho_n)\}, \text{dj}\{H(\sigma_n)\} \} \leq \text{dj}\{H(\rho_n + \sigma_n)\} \leq \text{dj}\{H(\rho_n)\} + \text{dj}\{H(\sigma_n)\}.$$

A strengthened version of Corollary 6 is obtained in Section 4.2 (Cor.9).

Proposition 3 and property (17) show that the loss of entropy does not increase under action of quantum operations with bounded Choi rank.

Corollary 7. *Let $\{\rho_n\} \subset \mathfrak{T}_+(\mathcal{H})$ be a sequence converging to an operator ρ_0 and $\{\Phi_n\}$ a sequence of quantum operations with bounded Choi rank such that the sequence $\{\Phi_n(\rho_n)\}$ converges to the operator $\Phi_0(\rho_0)$. Then*

$$\text{dj}\{H(\Phi_n(\rho_n))\} \leq \text{dj}\{H(\rho_n)\}. \quad (18)$$

In particular, (18) holds if $\{\Phi_n\}$ is a sequence of quantum operations with bounded Choi rank strongly converging to the operation Φ_0 (see Sect.5.2).

3.4 States with bounded energy

Let H be a positive unbounded operator in a Hilbert space \mathcal{H} which will be treated as a Hamiltonian of a quantum system associated with the space \mathcal{H} . Then

$$\mathcal{K}_{H,E} \doteq \{ \rho \in \mathfrak{S}(\mathcal{H}) \mid \text{Tr} H \rho \leq E \}$$

is the set of states with mean energy not exceeding E (here $\text{Tr} H\rho$ is defined as a limit of the nondecreasing sequence $\{\text{Tr} P_n H\rho\}$ of positive numbers, where P_n is the spectral projector of H corresponding to the interval $[0, n]$).

It is well known (cf.[3]) that the von Neumann entropy is continuous on the set $\mathcal{K}_{H,E}$ if

$$\text{Tr} e^{-\lambda H} < +\infty \quad \text{for all } \lambda > 0. \quad (19)$$

Recently Winter obtained explicit continuity bounds for the von Neumann entropy on the set $\mathcal{K}_{H,E}$ in this case [4]. In fact, (19) is a necessary and sufficient condition of continuity of the entropy on the set $\mathcal{K}_{H,E}$ if E is greater than the minimal energy level of H (this follows from Proposition 4 below). So, dealing with Hamiltonians not satisfying condition (19) we have to take into account discontinuity jumps of the von Neumann entropy (and of the related quantities) on the set $\mathcal{K}_{H,E}$.

Introduce the parameter

$$g(H) \doteq \inf\{\lambda > 0 \mid \text{Tr} e^{-\lambda H} < +\infty\},$$

which is assumed to be $+\infty$ if $\text{Tr} e^{-\lambda H} = +\infty$ for all $\lambda > 0$.

If $g(H) < +\infty$ then the operator H has discrete spectrum of finite multiplicity, i.e. it can be represented as follows

$$H = \sum_{k=0}^{+\infty} E_k |k\rangle\langle k|, \quad E_k \leq E_{k+1}, \quad (20)$$

where $\{|k\rangle\}$ is an orthonormal basic and $\{E_k\}$ is a nondecreasing sequence of eigenvalues (energy levels) of H . Since $g(H) = \limsup_k E_k^{-1} \log k$, the inequality $0 < g(H) < +\infty$ means the logarithmic growth of the sequence $\{E_k\}$. For any $\lambda > g(H)$ one can introduce the state $\sigma_\lambda = [\text{Tr} e^{-\lambda H}]^{-1} e^{-\lambda H}$. Then we have the identity

$$H(\rho) + H(\rho \parallel \sigma_\lambda) = \lambda \text{Tr} H\rho + C, \quad C = \log[\text{Tr} e^{-\lambda H}], \quad (21)$$

valid for any state ρ , which shows that the entropy is bounded on the set $\mathcal{K}_{H,E}$ for any $E > 0$.

If $g(H) = +\infty$ then the entropy is not bounded (and hence is not finite⁷) on the set $\mathcal{K}_{H,E}$ if $E > \inf_{\|\varphi\|=1} \langle \varphi | H | \varphi \rangle$.⁸

⁷It is easy to show that finiteness of the entropy on the closed convex set guarantees its boundedness on this set (see the proof of Theorem 1 in [18]).

⁸This can be shown by noting first that boundedness of the entropy on the set $\mathcal{K}_{H,E}$ implies that H has discrete spectrum of finite multiplicity and then by using the sequence of states (24) from the proof of Proposition 4 below.

Assume that the Hamiltonian H has form (20). For any state ρ introduce its rearrangement ρ^\downarrow corresponding to the basis $\{|k\rangle\}$ as (cf. [14])

$$\rho^\downarrow = \sum_{k=0}^{+\infty} \lambda_k |k\rangle \langle k|,$$

where $\{\lambda_k\}$ is the sequence of eigenvalues of ρ taken in decreasing order. It is clear that $H(\rho^\downarrow) = H(\rho)$. By using Ky Fan's Maximum Principle it is easy to show (see the proof of Lemma IV.9 in [14]) that

$$\text{Tr} H \rho^\downarrow \leq \text{Tr} H \rho. \quad (22)$$

Mirsky's inequality implies $\|\rho^\downarrow - \sigma^\downarrow\|_1 \leq \|\rho - \sigma\|_1$ [15]. So, the map $\rho \mapsto \rho^\downarrow$ is continuous. Hence the functions $\rho \mapsto E_H(\rho) \doteq \text{Tr} H \rho$ and $\rho \mapsto E_H(\rho^\downarrow)$ are lower semicontinuous on $\mathfrak{S}(\mathcal{H})$ and for a given sequence $\{\rho_n\} \subset \mathcal{K}_{H,E}$ converging to a state ρ_0 their discontinuity jumps are characterised by the nonnegative values

$$\text{dj}\{E_H(\rho_n)\} \doteq \limsup_{n \rightarrow +\infty} E_H(\rho_n) - E_H(\rho_0) \leq E$$

and

$$\text{dj}\{E_H(\rho_n^\downarrow)\} \doteq \limsup_{n \rightarrow +\infty} E_H(\rho_n^\downarrow) - E_H(\rho_0^\downarrow) \leq E,$$

where the last inequality follows from (22).

Proposition 4. *Let H be a positive operator and $E > E_0 \doteq \inf_{\|\varphi\|=1} \langle \varphi | H | \varphi \rangle$.*

A) *If $g(H) < +\infty$ then*

$$\text{dj}\{H(\rho_n)\} \leq g(H) \text{dj}\{E_H(\rho_n^\downarrow)\} \leq g(H) \text{dj}\{E_H(\rho_n)\} \leq g(H)(E - E_0) \quad (23)$$

for any converging sequence $\{\rho_n\} \subset \mathcal{K}_{H,E}$. All the bounds in (23) are sharp.

B) *In general case*

$$\sup_{\{\rho_n\} \subset \mathcal{K}_{H,E}} \text{dj}\{H(\rho_n)\} = g(H)(E - E_0) \leq +\infty,$$

where the supremum is over all converging sequences $\{\rho_n\} \subset \mathcal{K}_{H,E}$.

Remark 2. The first two inequalities in (23) are valid for arbitrary converging sequence $\{\rho_n\}$ if we assume that $\text{dj}\{E_H(\rho_n)\} = +\infty$ in the case $E_H(\rho_0) = +\infty$.

Proof. A) Assume that the operator H has form (20). Since $H(\rho_n^\downarrow) = H(\rho_n)$ for all n , equality (21) and the lower semicontinuity of the relative entropy imply, by Lemma 2, validity of the first inequality in (23) with $g(H)$ replaced by any $\lambda > g(H)$.

To prove the second one it suffices, by Lemma 2, to show that the non-negative function $f(\rho) \doteq E_H(\rho) - E_H(\rho^\downarrow)$ is lower semicontinuous on $\mathcal{K}_{H,E}$. Let

$$f_m(\rho) \doteq \text{Tr} H_m(\rho - \rho^\downarrow), \quad \text{where} \quad H_m = \sum_{k=0}^{m-1} E_k |k\rangle\langle k| + E_m \sum_{k=m}^{+\infty} |k\rangle\langle k|.$$

Since H_m is a bounded operator and the map $\rho \mapsto \rho^\downarrow$ is continuous, the function f_m is continuous on $\mathfrak{S}(\mathcal{H})$ for all m . Since $D_m \doteq H - H_m$ is an operator of the form (20), inequality (22) holds with H replaced by D_m and hence

$$f(\rho) - f_m(\rho) = \text{Tr} D_m(\rho - \rho^\downarrow) \geq 0$$

for any state ρ . It is easy to see that $f_m(\rho)$ tends to $f(\rho)$ for any state ρ with finite $E_H(\rho)$. Thus, the function f coincides on $\mathcal{K}_{H,E}$ with the least upper bound of the sequence $\{f_m\}$ of continuous functions.

The third inequality in (23) is obvious.

To show that all the bounds in (23) are sharp consider the sequence of states

$$\rho_n = \rho_n^\downarrow = (1 - q_n)|0\rangle\langle 0| + q_n n^{-1} \sum_{k=1}^n |k\rangle\langle k|, \quad (24)$$

where $\{q_n = (E - E_0)(n^{-1} \sum_{k=1}^n E_k - E_0)^{-1}\}$ is a sequence of positive numbers converging to zero (we assume that n is sufficiently large so that $q_n \leq 1$). The sequence $\{\rho_n\}$ lies in $\mathcal{K}_{H,E}$ (since $\text{Tr} H \rho_n = E$) and converges to the pure state $|0\rangle\langle 0|$. By concavity of the entropy we have

$$H(\rho_n) \geq q_n \log n = \frac{(E - E_0) \log n}{n^{-1} \sum_{k=1}^n E_k - E_0} \geq \frac{(E - E_0) \log n}{E_n - E_0}.$$

Since $\text{dj}\{E_H(\rho_n)\} = E - E_0$, to complete the proof of part A it suffices to note that $\limsup_n \log n(E_n - E_0)^{-1} = g(H)$.

B) We have only to prove the existence of a converging sequence $\{\rho_n\}$ such that $\text{dj}\{H(\rho_n)\} = +\infty$ in the case $g(H) = +\infty$. By the remark before

the proposition in this case there is a state $\sigma \in \mathcal{K}_{H,E}$ such that $H(\sigma) = +\infty$. The sequence consisting of the states $\rho_n = n^{-1}\sigma + (1 - n^{-1})\rho_0$, where ρ_0 is any pure state in $\mathcal{K}_{H,E}$, possesses the required property. \square

4 Estimates for discontinuity of some information quantities

In this section we show that discontinuity jumps of many information characteristics of composite quantum states are upper bounded by discontinuity jump of one of the marginal entropies (with a corresponding coefficient).

4.1 Quantum mutual information and conditional entropy

Quantum mutual information of a state ω_{AB} of an infinite-dimensional bipartite quantum system is defined as follows (cf.[19])

$$I(A:B)_\omega = H(\omega_{AB} \| \omega_A \otimes \omega_B).$$

We will use the homogeneous extension of this quantity to positive trace-class operators

$$I(A:B)_\omega \doteq [\text{Tr}\omega] I(A:B)_{\frac{\omega}{\text{Tr}\omega}}, \quad \omega \in \mathfrak{T}_+(\mathcal{H}_{AB}).$$

Basic properties of the relative entropy show that $\omega \mapsto I(A:B)_\omega$ is a lower semicontinuous function on the cone $\mathfrak{T}_+(\mathcal{H}_{AB})$ taking values in $[0, +\infty]$. It is easy to show that (cf.[20])

$$I(A:B)_\omega \leq 2 \min \{H(\omega_A), H(\omega_B)\}. \quad (25)$$

By the lower semicontinuity of quantum mutual information its discontinuity for a given sequence $\{\omega_{AB}^n\} \subset \mathfrak{T}_+(\mathcal{H}_{AB})$ converging to an operator $\omega_{AB}^0 \in \mathfrak{T}_+(\mathcal{H}_{AB})$ is characterised by the nonnegative value

$$\text{dj}\{I(A:B)_{\omega^n}\} \doteq \limsup_{n \rightarrow +\infty} I(A:B)_{\omega^n} - I(A:B)_{\omega^0}$$

which can be called *mutual information loss* corresponding to this sequence (it is assumed as usual that $\text{dj}\{I(A:B)_{\omega^n}\} = +\infty$ if $I(A:B)_{\omega^0} = +\infty$).

The following theorem is essentially used below.

Theorem 1. *Let $\{\omega_{AB}^n\} \subset \mathfrak{T}_+(\mathcal{H}_{AB})$ be a sequence converging to an operator ω_{AB}^0 and $\Phi : A \rightarrow C$, $\Psi : B \rightarrow D$ be quantum operations. Then*

$$\text{dj}\{I(C:D)_{\Phi \otimes \Psi(\omega_{AB}^n)}\} \leq \text{dj}\{I(A:B)_{\omega^n}\} \leq 2 \min\{\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_B^n)\}\}.$$

Example 1 in [5] shows that $\text{dj}\{I(A:B)_{\omega^n}\}$ may vanish despite positivity of $\min\{\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_B^n)\}\}$. On the other hand, by considering sequences of pure states we see that this upper bound is sharp.

The first inequality in Theorem 1 means that *discontinuity jumps of quantum mutual information do not increase under action of local operations*. So, it generalizes Theorem 1B in [5] stating that local continuity of quantum mutual information is preserved by local operations.

Proof. To prove the second inequality of the theorem we will use the identity

$$I(A:B)_\omega + I(A:C)_\omega = 2H(\omega_A) \quad (26)$$

valid for any 1-rank operator $\omega \in \mathfrak{T}_+(\mathcal{H}_{ABC})$ (with possible value $+\infty$ in the both sides). If $H(\omega_A)$, $H(\omega_B)$ and $H(\omega_C)$ are finite then (26) is easily verified by noting that $H(\omega_A) = H(\omega_{BC})$, $H(\omega_B) = H(\omega_{AC})$ and $H(\omega_C) = H(\omega_{AB})$. In general case (26) can be proved by approximation (see the proof of Theorem 1 in [5, the Appendix]).

It suffices to prove the inequality $\text{dj}\{I(A:B)_{\omega^n}\} \leq 2\text{dj}\{H(\omega_A^n)\}$ assuming that $H(\omega_A^0) < +\infty$. By Lemma 1 there is a sequence $\{\tilde{\omega}_{ABC}^n\}$ of 1-rank operators in $\mathfrak{T}_+(\mathcal{H}_{ABC})$ converging to an operator $\tilde{\omega}_{ABC}^0$ such that $\tilde{\omega}_{AB}^n = \omega_{AB}^n$ for all $n \geq 0$. By Lemma 2 identity (26) and the lower semicontinuity of the function $\omega_{ABC} \mapsto I(A:C)_\omega$ imply the required inequality.

To prove the first inequality of the theorem it suffices to show that

$$\text{dj}\{I(C:B)_{\Phi \otimes \text{Id}_B(\omega_{AB}^n)}\} \leq \text{dj}\{I(A:B)_{\omega^n}\} \quad (27)$$

for any quantum operation $\Phi : A \rightarrow C$. We will use the identity (chain rule)

$$I(A:B)_\omega + I(B:C|A)_\omega = I(AC:B)_\omega,$$

where $I(B:C|A)_\omega$ is the conditional mutual information extended to the cone $\mathfrak{T}_+(\mathcal{H}_{ABC})$ (see Section 4.3 below). By Lemma 2 this identity and the lower semicontinuity of $I(B:C|A)_\omega$ (stated in [5, Th.2]) imply

$$\text{dj}\{I(A:B)_{\omega^n}\} \leq \text{dj}\{I(AC:B)_{\omega^n}\} \quad (28)$$

for any converging sequence $\{\omega_{ABC}^n\} \subset \mathfrak{T}_+(\mathcal{H}_{ABC})$. By using the Stinespring representation (3) one can show that (28) implies (27) for any quantum channel Φ .

If Φ is a trace non-preserving operation then consider the channel $\Phi' = \Phi \oplus \Delta$ from A to $C' = C \oplus C^c$, where $\Delta(\rho) = [\text{Tr}\rho - \text{Tr}\Phi(\rho)]\sigma$ is a quantum operation from A to C^c determined by a fixed state $\sigma \in \mathfrak{S}(\mathcal{H}_{C^c})$. We have

$$\begin{aligned} I(C':B)_{\Psi \otimes \text{Id}_B(\omega_{AB})} &= I(C:B)_{\tilde{\omega}} + H(\tilde{\omega}_B \| \lambda \omega_B) \\ &+ H(\Delta \otimes \text{Id}_B(\omega_{AB}) \| \Delta(\omega_A) \otimes \omega_B), \end{aligned} \quad (29)$$

where $\tilde{\omega}_{CB} = \Phi \otimes \text{Id}_B(\omega_{AB})$ and $\lambda = \text{Tr}\tilde{\omega}_{CB}$ (see the proof of Th.1B in [5]).

Since all the summands in the right hand side of (29) are lower semicontinuous functions, Lemma 2 implies

$$\text{dj}\{I(C:B)_{\Phi \otimes \text{Id}_B(\omega_{AB}^n)}\} \leq \text{dj}\{I(C':B)_{\Phi' \otimes \text{Id}_B(\omega_{AB}^n)}\} \leq \text{dj}\{I(A:B)_{\omega^n}\},$$

where the second inequality holds, since Φ' is a channel. \square

The quantum conditional entropy

$$H(A|B)_\omega = H(\omega_{AB}) - H(\omega_B) \quad (30)$$

can be extended to the convex set $\mathfrak{S}_A \doteq \{\omega_{AB} \mid H(\omega_A) < +\infty\}$ containing states with $H(\omega_{AB}) = H(\omega_B) = +\infty$ by the formula

$$H(A|B)_\omega = H(\omega_A) - I(A:B)_\omega \quad (31)$$

preserving all basic properties of the conditional entropy [21]. Upper bound (25) shows that $H(A|B)_\omega$ takes values in the interval $[-H(\omega_A), H(\omega_A)]$.

The conditional entropy is not upper or lower semicontinuous.⁹ So, its discontinuity jumps for a given sequence $\{\omega_{AB}^n\} \subset \mathfrak{S}_A$ converging to a state $\omega_{AB}^0 \in \mathfrak{S}_A$ can be characterised by two nonnegative values

$$\text{dj}^\downarrow\{H(A|B)_{\omega^n}\} \doteq \max\left\{\limsup_{n \rightarrow +\infty} H(A|B)_{\omega^n} - H(A|B)_{\omega^0}, 0\right\}$$

and

$$\text{dj}^\uparrow\{H(A|B)_{\omega^n}\} \doteq \max\left\{H(A|B)_{\omega^0} - \liminf_{n \rightarrow +\infty} H(A|B)_{\omega^n}, 0\right\}$$

⁹By Corollary 4 in Sec.3.2 the conditional entropy is lower semicontinuous on the set of separable states with finite entropy.

describing respectively the maximal loss and the maximal gain of the conditional entropy corresponding to this sequence.

Corollary 8. *Let $\{\omega_{AB}^n\}$ be a sequence converging to a state ω_{AB}^0 such that $H(\omega_A^n) < +\infty$ for all $n \geq 0$. Then*

$$\text{dj}^\downarrow\{H(A|B)_{\omega^n}\} \leq \min\{\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_{AB}^n)\}\}, \quad (32)$$

$$\text{dj}^\uparrow\{H(A|B)_{\omega^n}\} \leq \min\{2\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_B^n)\}\}.$$

If $\{H(\omega_A^n)\}$ is a converging sequence then the factor 2 in the last inequality can be removed. If $\omega_{AB}^n, \omega_{AB}^0$ are separable states with finite entropy then $\text{dj}^\uparrow\{H(A|B)_{\omega^n}\} = 0$.

Proof. Inequalities (32) and $\text{dj}^\uparrow\{H(A|B)_{\omega^n}\} \leq \text{dj}\{H(\omega_B^n)\}$ are derived from (30) and (31) by using Lemma 2 and the lower semicontinuity of $H(\omega_{AB})$, $H(\omega_B)$ and $I(A:B)_\omega$.

If $\text{dj}^\uparrow\{H(A|B)_{\omega^n}\} > 0$ then (31) implies

$$\text{dj}^\uparrow\{H(A|B)_{\omega^n}\} \leq \left[\limsup_{n \rightarrow +\infty} I(A:B)_{\omega^n} - I(A:B)_{\omega^0} \right] - \left[\liminf_{n \rightarrow +\infty} H(\omega_A^n) - H(\omega_A^0) \right].$$

So, the inequality $\text{dj}^\uparrow\{H(A|B)_{\omega^n}\} \leq 2\text{dj}\{H(\omega_A^n)\}$ follow from Theorem 1 and the lower semicontinuity of $H(\omega_A)$. If $\{H(\omega_A^n)\}$ is a converging sequence then $\liminf_{n \rightarrow +\infty} H(\omega_A^n) - H(\omega_A^0) = \text{dj}\{H(\omega_A^n)\}$.

The last assertion follows from the lower semicontinuity of the conditional entropy on the set of separable states with finite entropy (Cor.4 in Sec.3.2). \square

4.2 The Holevo quantity of ensemble of quantum states

The Holevo quantity of an ensemble $\{\pi_i, \rho_i\}$ of quantum states is defined as

$$\chi(\{\pi_i, \rho_i\}) \doteq \sum_i \pi_i H(\rho_i \| \bar{\rho}) = H(\bar{\rho}) - \sum_i \pi_i H(\rho_i), \quad \bar{\rho} = \sum_i \pi_i \rho_i,$$

where the second formula is valid if $H(\bar{\rho}) < +\infty$. It plays a basic role in analysis of information properties of quantum systems and channels [2, 8].

We will say that a sequence $\{\{\pi_i^n, \rho_i^n\}_i\}_n$ of ensembles converges to an ensemble $\{\pi_i^0, \rho_i^0\}$ if

$$\lim_{n \rightarrow \infty} \pi_i^n = \pi_i^0 \text{ for all } i \text{ and } \lim_{n \rightarrow \infty} \rho_i^n = \rho_i^0 \text{ for all } i \text{ s.t. } \pi_i^0 \neq 0. \quad (33)$$

The lower semicontinuity of the relative entropy implies lower semicontinuity of the Holevo quantity with respect to this convergence. So, its discontinuity for a sequence $\{\{\pi_i^n, \rho_i^n\}_i\}_n$ converging to an ensemble $\{\pi_i^0, \rho_i^0\}$ is characterised by the nonnegative value

$$\text{dj}\{\chi(\{\pi_i^n, \rho_i^n\})\} \doteq \limsup_{n \rightarrow +\infty} \chi(\{\pi_i^n, \rho_i^n\}) - \chi(\{\pi_i^0, \rho_i^0\})$$

which can be called *loss of the Holevo quantity* corresponding to this sequence (it is assumed that $\text{dj}\{\chi(\{\pi_i^n, \rho_i^n\})\} = +\infty$ if $\chi(\{\pi_i^0, \rho_i^0\}) = +\infty$).

Proposition 5. *Let $\{\{\pi_i^n, \rho_i^n\}_{i=1}^m\}$ be a sequence of ensembles consisting of $m \leq +\infty$ states converging to an ensemble $\{\{\pi_i^0, \rho_i^0\}_{i=1}^m\}$. Then*

$$\text{dj}\{\chi(\{\pi_i^n, \rho_i^n\}_{i=1}^m)\}_n \leq \min\{\text{dj}\{H(\bar{\rho}_n)\}, 2\text{dj}\{S(\bar{\pi}_n)\}\},$$

where $\bar{\rho}_n \doteq \sum_{i=1}^m \pi_i^n \rho_i^n$ and $\bar{\pi}_n$ is the probability distribution $\{\pi_i^n\}_{i=1}^m$.

If $\lim_{n \rightarrow +\infty} S(\bar{\pi}_n) = S(\bar{\pi}_0) < +\infty$ (in particular, if $m < +\infty$) then

$$\text{dj}\left\{H\left(\sum_{i=1}^m \pi_i^n \rho_i^n\right)\right\}_n = \text{dj}\left\{\sum_{i=1}^m \pi_i^n H(\rho_i^n)\right\}_n.$$

Proof. To prove the inequality $\text{dj}\{\chi(\{\pi_i^n, \rho_i^n\}_{i=1}^m)\}_n \leq \text{dj}\{H(\bar{\rho}_n)\}$ we may assume that $H(\bar{\rho}_n)$ is finite for all n . So, we have

$$\chi(\{\pi_i^n, \rho_i^n\}_{i=1}^m) + \sum_{i=1}^m \pi_i^n H(\rho_i^n) = H(\bar{\rho}_n) \quad \forall n. \quad (34)$$

Thus, the required inequality follows from Lemma 2 and the lower semicontinuity of the second term in (34) with respect to the convergence (33).

To prove the inequality $\text{dj}\{\chi(\{\pi_i^n, \rho_i^n\}_{i=1}^m)\}_n \leq 2\text{dj}\{S(\bar{\pi}_n)\}$ assume that $\mathcal{H}_A = \mathcal{H}$ and $\mathcal{H}_B = \mathbb{C}^m$. It is easy to see that $\chi(\{\pi_i^n, \rho_i^n\}_{i=1}^m) = I(A:B)_{\omega^n}$ for each $n \geq 0$, where

$$\omega_{AB}^n = \sum_{i=1}^m \pi_i^n \rho_i^n \otimes |i\rangle\langle i| \quad (35)$$

is a state in $\mathfrak{S}(\mathcal{H}_{AB})$ determined by a basis $\{|i\rangle\}$ in \mathcal{H}_B . Since $H(\omega_B^n) = S(\bar{\pi}_n)$ for all n , to obtain the required inequality from Theorem 1 it suffices to show convergence of the sequence $\{\omega_{AB}^n\}$ to the state ω_{AB}^0 . This can be done by noting that (33) implies convergence of the sequence $\{\omega_{AB}^n\}$ to the state ω_{AB}^0 in the weak operator topology and by using the result from [22].

The second assertion of the proposition follows from the first one. \square

Proposition 5B implies the following strengthened version of Corollary 6.

Corollary 9. *Let $\{\rho_n^1\}_n, \dots, \{\rho_n^m\}_n$ be sequences of operators in $\mathfrak{T}_+(\mathcal{H})$ converging to operators $\rho_0^1, \dots, \rho_0^m$, where $m \leq +\infty$. The equality*

$$\text{dj} \left\{ H \left(\sum_{k=1}^m \rho_n^k \right) \right\}_n = \text{dj} \left\{ \sum_{k=1}^m H(\rho_n^k) \right\}_n \quad (36)$$

holds if

$$\lim_{n \rightarrow \infty} \sum_{k=1}^m \text{Tr} \rho_n^k = \sum_{k=1}^m \text{Tr} \rho_0^k < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} S(\{\text{Tr} \rho_n^k\}_{k=1}^m) = S(\{\text{Tr} \rho_0^k\}_{k=1}^m) < \infty,$$

where $S(\{x_k\}) \doteq \sum_k \eta(x_k) - \eta(\sum_k x_k)$ is the homogeneous extension of the Shannon entropy to the positive cone of ℓ_1 . In particular, relation (36) holds if $m < +\infty$.

4.3 Conditional mutual information

The conditional mutual information of a state ω_{ABC} of a tripartite finite-dimensional system is defined as follows

$$I(A:C|B)_\omega \doteq H(\omega_{AB}) + H(\omega_{BC}) - H(\omega_{ABC}) - H(\omega_B). \quad (37)$$

This quantity plays important role in quantum information theory [23, 24], its nonnegativity is a basic result well known as *strong subadditivity of von Neumann entropy* [1].

In infinite dimensions formula (37) may contain the uncertainty " $\infty - \infty$ ". Nevertheless the conditional mutual information can be defined for any state ω_{ABC} by one of the equivalent expressions

$$I(A:C|B)_\omega = \sup_{P_A} [I(A:BC)_{Q_A \omega Q_A} - I(A:B)_{Q_A \omega Q_A}], \quad Q_A = P_A \otimes I_{BC}, \quad (38)$$

$$I(A:C|B)_\omega = \sup_{P_C} [I(AB:C)_{Q_C \omega Q_C} - I(B:C)_{Q_C \omega Q_C}], \quad Q_C = P_C \otimes I_{AB}, \quad (39)$$

where the suprema are over all finite rank projectors $P_A \in \mathfrak{B}(\mathcal{H}_A)$ and $P_C \in \mathfrak{B}(\mathcal{H}_C)$ correspondingly [5].

It is shown in [5, Th.2] that expressions (38) and (39) define a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{ABC})$ possessing all basic properties of conditional mutual information valid in finite dimensions. If one of the marginal entropies $H(\omega_A)$, $H(\omega_C)$ and $H(\omega_B)$ is finite then the above extension is given respectively by the explicit formula¹⁰

$$I(A:C|B)_\omega = I(A:BC)_\omega - I(A:B)_\omega, \quad (40)$$

$$I(A:C|B)_\omega = I(AB:C)_\omega - I(B:C)_\omega \quad (41)$$

and

$$I(A:C|B)_\omega = I(A:C)_\omega - I(A:B)_\omega - I(B:C)_\omega + I(AC:B)_\omega. \quad (42)$$

Since $\omega_{ABC} \mapsto I(A:C|B)_\omega$ is a lower semicontinuous function, its discontinuity for a given sequence $\{\omega_{ABC}^n\}$ of states converging to a state ω_{ABC}^0 with finite $I(A:C|B)_{\omega^0}$ is characterised by the nonnegative value

$$\text{dj}\{I(A:C|B)_{\omega^n}\} \doteq \limsup_{n \rightarrow +\infty} I(A:C|B)_{\omega^n} - I(A:C|B)_{\omega^0}.$$

Proposition 6. *For an arbitrary sequence $\{\omega_{ABC}^n\}$ of states converging to a state ω_{ABC}^0 the following inequalities hold*

$$\text{dj}\{I(A:C|B)_{\omega^n}\} \leq 2 \min \{ \text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_C^n)\}, \text{dj}\{H(\omega_{AB}^n)\}, \text{dj}\{H(\omega_{BC}^n)\} \},$$

$$\text{dj}\{I(A:C|B)_{\omega^n}\} \leq \text{dj}\{I(A:C)_{\omega^n}\} + 2 \min \{ \text{dj}\{H(\omega_B^n)\}, \text{dj}\{H(\omega_{ABC}^n)\} \}.$$

Proof. The first inequality is derived from representations (40) and (41) by using Lemma 2, the lower semicontinuity of the quantum mutual information and Theorem 1. The inequality

$$\text{dj}\{I(A:C|B)_{\omega^n}\} \leq \text{dj}\{I(A:C)_{\omega^n}\} + 2\text{dj}\{H(\omega_B^n)\}$$

is derived by the same way from representation (42).

To prove the inequality $\text{dj}\{I(A:C|B)_{\omega^n}\} \leq \text{dj}\{I(A:C)_{\omega^n}\} + 2\text{dj}\{H(\omega_{ABC}^n)\}$ note that Lemma 1 implies existence of a sequence $\{\tilde{\omega}_{ABCD}^n\}$ of pure states converging to a state $\tilde{\omega}_{ABCD}^0$ such that $\tilde{\omega}_{ABC}^n = \omega_{ABC}^n$ for all $n \geq 0$. We have (cf.[23])

$$I(A:C|B)_{\omega^n} = I(A:C|D)_{\tilde{\omega}^n} \quad \text{and} \quad H(\omega_{ABC}^n) = H(\tilde{\omega}_D^n) \quad \text{for all } n \geq 0.$$

So, the required inequality is derived from representation (42) applied to $I(A:C|D)$ by using Lemma 2, the lower semicontinuity of the quantum mutual information and Theorem 1. \square

¹⁰The correctness of these formulas follows from upper bound (25).

4.4 Several entanglement measures

In this subsection we will obtain estimates for discontinuity jumps of the infinite-dimensional versions of squashed entanglement, c-squashed entanglement, entanglement of formation and of their regularizations.

The *squashed entanglement* E_{sq} of a state ω_{AB} of a finite dimensional bipartite system is defined as follows

$$E_{sq}(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E), \quad (43)$$

where $I(A:B|E)_\omega$ is the conditional mutual information defined by (37) and the infimum is over all extensions ω_{ABE} of the state ω_{AB} [25, 26]. The squashed entanglement is an unique known entanglement measure possessing all basic properties of an entanglement measure including additivity and monogamy [25, 27, 37].

Possible generalizations of squashed entanglement to states of infinite-dimensional bipartite system are considered in [28], where it is shown that it can be unambiguously defined on the set

$$\mathfrak{S}_* \doteq \{\omega_{AB} \mid \min\{H(\omega_A), H(\omega_B), H(\omega_{AB})\} < +\infty\} \quad (44)$$

by the same formula (43), where $I(A:B|E)_\omega$ is the extended conditional mutual information described in the previous subsection, as a lower semicontinuous entanglement measure possessing all basic properties of the squashed entanglement valid in finite dimensions.

Remark 3. It is shown in [28] that any continuous finite-dimensional entanglement measure has an unique lower semicontinuous extension to the set of all infinite-dimensional bipartite states, but it is not clear how to prove coincidence of this "universal" extension with the quantity obtained by direct translation of the finite-dimensional definition. The above set \mathfrak{S}_* is the maximal set of states on which such coincidence is proved for the squashed entanglement (as well as for the c-squashed entanglement and for the entanglement of formation considered below). \square

The *c-squashed entanglement* E_{csq} of a state ω_{AB} of a finite-dimensional bipartite system is defined by the formula

$$E_{csq}(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE} \in \mathfrak{S}_c} I(A:B|E), \quad (45)$$

where \mathfrak{S}_c is the set of all extensions of ω_{AB} having the form

$$\omega_{ABE} = \sum_i \pi_i \omega_{AB}^i \otimes |i\rangle\langle i|_E. \quad (46)$$

This means that

$$E_{csq}(\omega_{AB}) = \inf_{\sum_i \pi_i \omega_{AB}^i = \omega_{AB}} \sum_i \pi_i I(A:B)_{\omega^i}, \quad (47)$$

where the infimum is over all ensembles $\{\pi_i, \omega_{AB}^i\}$ of states with the average state ω_{AB} [29, 30], i.e. E_{csq} is the convex hull (mixed convex roof) of the quantum mutual information.

An universal infinite-dimensional extension (mentioned in Remark 3) of the c-squashed entanglement is given by the formula

$$E_{csq}(\omega_{AB}) = \inf_{b(\mu) = \omega_{AB}} \int I(A:B)_\omega \mu(d\omega), \quad (48)$$

where the infimum is over all Borel probability measures on the set $\mathfrak{S}(\mathcal{H}_{AB})$ with the barycenter ω_{AB} .¹¹ Indeed, Proposition 1 in [31] and Corollary 1 in [31] imply lower semicontinuity of the right hand side of (48) and its coincidence with the right hand side of (47) for any state ω_{AB} with finite rank marginals.

By using Corollary 6 in [31] and upper bound (25) one can show that formulas (47) and (48) coincide for any state ω_{AB} in the set \mathfrak{S}_* defined in (44). So, representation (45) remains valid for any $\omega_{AB} \in \mathfrak{S}_*$ and hence E_{csq} is not less than E_{sq} on \mathfrak{S}_* . Global coincidence of (47) and (48) is an open question. By Proposition 1 in [28] it is equivalent to global lower semicontinuity of the right hand side of (47).

The *entanglement of formation* of a state ω_{AB} of a finite dimensional bipartite system is defined as follows

$$E_F(\omega_{AB}) = \inf_{\sum_i \pi_i \omega_{AB}^i = \omega_{AB}} \sum_i \pi_i H(\omega_A^i), \quad (49)$$

where the infimum is over all ensembles $\{\pi_i, \omega_{AB}^i\}$ of pure states with the average state ω_{AB} [32]. The entanglement of formation is one of the most important entanglement measures – it is the maximal convex continuous

¹¹The integral is well defined for any such μ due to the lower semicontinuity of $I(A:B)_\omega$.

function coinciding with the marginal entropy of a state on the set of pure bipartite states [30, 33].

An universal infinite-dimensional extension (mentioned in Remark 3) of the entanglement of formation is given by the formula

$$E_F(\omega_{AB}) = \inf_{b(\mu)=\omega_{AB}} \int H(\omega_A) \mu(d\omega), \quad (50)$$

where the infimum is over all Borel probability measures on the set $\text{ext } \mathfrak{S}(\mathcal{H}_{AB})$ of pure states with the barycenter ω_{AB} (see the end of Sect.3 in [28]). Similar to the case of E_{csq} formulas (49) and (50) coincide for any state ω_{AB} in the set \mathfrak{S}_* defined in (44). Global coincidence of (49) and (50) is a conjecture equivalent to global lower semicontinuity of the right hand side of (49).

Since the right hand side of (49) can be written as the right hand side of (45) with the set \mathfrak{S}_c replaced by its subset \mathfrak{S}_c^p consisting of all states (46) such that $\text{rank } \omega_{AB}^i = 1$ for all i (cf.[25]), we have

$$E_{sq}(\omega_{AB}) \leq E_{csq}(\omega_{AB}) \leq E_F(\omega_{AB})$$

for any state ω_{AB} in \mathfrak{S}_* . Examples showing that " $<$ " may hold in the above inequalities are presented in [24, 25]. Note also that E_{sq} , E_{csq} and E_F have the common continuity bound under the energy constraint on one subsystem provided the corresponding Hamiltonian satisfies condition (19). For E_{sq} and E_F this continuity bound is obtained in [28], the case of E_{csq} is considered similarly. This bound implies the asymptotic continuity of all these entanglement measures under the energy constraint on one subsystem.

In contrast to the squashed entanglement E_{sq} , the measures E_{csq} and E_F are nonadditive. To obtain additive measures consider the regularizations

$$E_{csq}^\infty(\omega_{AB}) \doteq \lim_{k \rightarrow \infty} k^{-1} E_{csq}(\omega_{AB}^{\otimes k}), \quad E_F^\infty(\omega_{AB}) \doteq \lim_{k \rightarrow \infty} k^{-1} E_F(\omega_{AB}^{\otimes k}).$$

In finite dimensions $E_F^\infty(\omega_{AB})$ coincides with the entanglement cost $E_C(\omega_{AB})$ – an operationally defined entanglement measure [34].

Since E_{sq} , E_{csq} and E_F are lower semicontinuous functions on the set \mathfrak{S}_* , discontinuity jumps of these functions for a given sequence $\{\omega_{AB}^n\} \subset \mathfrak{S}_*$ converging to a state $\omega_{AB}^0 \in \mathfrak{S}_*$ are characterised by the nonnegative values

$$\text{dj}\{E(\omega_{AB}^n)\} \doteq \limsup_{n \rightarrow +\infty} E(\omega_{AB}^n) - E(\omega_{AB}^0), \quad E = E_{sq}, E_{csq}, E_F, \quad (51)$$

where it is assumed as usual that $\text{dj}\{E(\omega_{AB}^n)\} = +\infty$ if $E(\omega_{AB}^0) = +\infty$.

Lower semicontinuity of the functions E_{csq}^∞ and E_F^∞ on the set \mathfrak{S}_* is conjectured but not proved.¹² Nevertheless, we can consider the values $\text{dj}\{E_{csq}^\infty(\omega_{AB}^n)\}$ and $\text{dj}\{E_F^\infty(\omega_{AB}^n)\}$ defined by formula (51) with $E = E_{csq}^\infty, E_F^\infty$ characterizing maximal loss of these functions for a given converging sequence $\{\omega_{AB}^n\} \subset \mathfrak{S}_*$.

Proposition 2 in [28] and Proposition 8 in [31] show that the functions E_{sq} and E_F are continuous on any subset of $\mathfrak{S}(\mathcal{H}_{AB})$ on which one of the marginal entropies $H(\omega_A)$ and $H(\omega_B)$ is continuous. The same condition is valid for E_{csq} . These observations are generalized in the following

Proposition 7. *For an arbitrary sequence $\{\omega_{AB}^n\} \subset \mathfrak{S}_*$ converging to a state $\omega_{AB}^0 \in \mathfrak{S}_*$ the following inequalities hold*

$$\begin{aligned} \text{dj}\{E(\omega_{AB}^n)\} &\leq \min\{\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_B^n)\}\}, & E = E_{sq}, E_{csq}, E_{csq}^\infty, E_F, E_F^\infty \\ \text{dj}\{E(\omega_{AB}^n)\} &\leq \frac{1}{2}\text{dj}\{I(A:B)_{\omega^n}\}, & E = E_{sq}, E_{csq}, E_{csq}^\infty. \end{aligned}$$

Proof. The first inequality for $E = E_F$ follows from Remark 5 in Section 5.2 below, since $E_F(\omega_{AB}) = \overline{\text{co}}H_\Phi(\omega_{AB})$, where $\Phi(\omega_{AB}) = \omega_A$, for any state $\omega_{AB} \in \mathfrak{S}_*$ [31].

The first inequality for $E = E_{csq}, E_{sq}$ follows from the second one and Theorem 1.

To prove the second inequality for $E = E_{sq}$ assume that $I(A:B)_{\omega^n} < +\infty$ for all n and consider the nonincreasing sequence of functions

$$E_{sq}^k(\omega_{AB}) = \frac{1}{2} \inf_{\omega_{ABE}} I(A:B|E)_\omega, \quad \dim \mathcal{H}_E \leq k$$

pointwise converging to the function E_{sq} on \mathfrak{S}_* [28, Lemma 4]. By Lemma 4 in Section 2 it suffices to show that $\text{dj}\{E_{sq}^k(\omega_{AB}^n)\} \leq \frac{1}{2}\text{dj}\{I(A:B)_{\omega^n}\}$ for all k .

Let \mathcal{H}_E^k be a k -dimensional Hilbert space and $\tilde{\omega}_{ABE}^0 \in \mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_E^k)$ be an extension of the state ω_{AB}^0 such that $E_{sq}^k(\omega_{AB}^0) \geq \frac{1}{2}I(A:B|E)_{\tilde{\omega}^0} - \varepsilon$.

¹²Recently Winter proved the continuity of E_F^∞ of the set $\mathfrak{S}(\mathcal{H}_{AB})$ if one of the systems A and B is finite-dimensional [4]. So, to prove the lower semicontinuity of E_F^∞ on the set \mathfrak{S}_* it suffices to show that $\sup_n \lambda_n E_F^\infty(\omega_{AB}^n) = E_F^\infty(\omega_{AB})$, where $\lambda_n = \text{Tr} P_A^n \otimes I_B \omega_{AB}$, $\omega_{AB}^n = \lambda_n^{-1} P_A^n \otimes I_B \omega_{AB} P_A^n \otimes I_B$, for all $\omega_{AB} \in \mathfrak{S}_*$ and some sequence $\{P_A^n\}$ of finite rank projectors strongly converging to the identity operator I_A .

By using Lemma 1 it is easy to show existence of a sequence $\{\tilde{\omega}_{ABE}^n\}$ in $\mathfrak{S}(\mathcal{H}_{AB} \otimes \mathcal{H}_E^k)$ converging to the state $\tilde{\omega}_{ABE}^0$ such that $\tilde{\omega}_{AB}^n = \omega_{AB}^n$ for all n .

Since $E_{sq}^k(\omega_{AB}^n) \leq \frac{1}{2}I(A:B|E)_{\tilde{\omega}^n}$ for all n , the second inequality in Proposition 6 implies

$$\text{dj}\{E_{sq}^k(\omega_{AB}^n)\} \leq \frac{1}{2}\text{dj}\{I(A:B|E)_{\tilde{\omega}^n}\} + \varepsilon \leq \frac{1}{2}\text{dj}\{I(A:B)_{\omega^n}\} + \varepsilon.$$

The case $E = E_{csq}$ is considered similarly. By using Corollary 6 in [31] and upper bound (25) one can show that the sequence of functions

$$E_{csq}^k(\omega_{AB}) \doteq \frac{1}{2} \inf_{\omega_{ABE} \in \mathfrak{S}_c^k} I(A:B|E)_\omega = \inf_{\sum_{i=1}^k \pi_i \omega_{AB}^i = \omega_{AB}} \sum_{i=1}^k \pi_i I(A:B)_{\omega^i}$$

where \mathfrak{S}_c^k is the subset of \mathfrak{S}_c consisting of states (46) with number of summands $\leq k$, pointwise converges to the function E_{csq} on \mathfrak{S}_* . It suffices only to show existence of a sequence $\{\tilde{\omega}_{ABE}^n\} \subset \mathfrak{S}_c^k$ converging to a given state $\tilde{\omega}_{ABE}^0 \in \mathfrak{S}_c$ such that $\tilde{\omega}_{AB}^n = \omega_{AB}^n$ for all n . But this follows from stability of the set $\mathfrak{S}(\mathcal{H}_{AB})$ [17], since it implies that for an ensemble $\{\pi_i^0, \rho_i^0\}_{i=1}^k$ with the average state ρ_0 and a sequence $\{\rho_n\}$ converging to the state ρ_0 there exists a sequence $\{\{\pi_i^n, \rho_i^n\}_{i=1}^k\}_n$ of ensembles such that $\sum_{i=1}^k \pi_i^n \rho_i^n = \rho_n$ converging to the ensemble $\{\pi_i^0, \rho_i^0\}_{i=1}^k$ in the sense of (33).

Consider the cases $E = E_{csq}^\infty, E_F^\infty$. Since the functions E_{csq} and E_F are subadditive for tensor product states, the functions E_{csq}^∞ and E_F^∞ are pointwise limits of the non-increasing sequences of functions

$$E_{csq}^k(\omega_{AB}) \doteq k^{-1} E_{csq}(\omega_{AB}^{\otimes k}), \quad E_F^k(\omega_{AB}) \doteq k^{-1} E_F(\omega_{AB}^{\otimes k}).$$

By Lemma 4 in Section 2 to prove the required estimates for $\text{dj}\{E_{csq}^\infty(\omega_{AB}^n)\}$ and $\text{dj}\{E_F^\infty(\omega_{AB}^n)\}$ it suffices to show that

$$\text{dj}\{E(\omega_{AB}^n)\} \leq \min \{\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_B^n)\}\}, \quad E = E_{csq}^k, E_F^k$$

and that

$$\text{dj}\{E_{csq}^k(\omega_{AB}^n)\} \leq \frac{1}{2}\text{dj}\{I(A:B)_{\omega^n}\}$$

for all k . But these relations follow from the same relations with $k = 1$ proved before due to the additivity of the von Neumann entropy and of the quantum mutual information. \square

4.5 The Henderson-Vedral measure of classical correlations and quantum discord

To describe classical component of correlation of a state ω_{AB} of a finite dimensional bipartite system Henderson and Vedral introduced in [35] the notion of a measure of classical correlations (as a function satisfying several basic requirements). They also proposed an example of such measure defined as follows

$$C_B(\omega_{AB}) = \sup_{\{M_i\}} \left[H(\omega_A) - \sum_i \pi_i H(\omega_A^i) \right], \quad (52)$$

where the supremum is taken over all measurements (POVM) $\{M_i\}$ applied to the system B , $\pi_i = \text{Tr}[(I_A \otimes M_i)\omega_{AB}]$ is the probability of the outcome i , $\omega_A^i = \pi_i^{-1} \text{Tr}_B[(I_A \otimes M_i)\omega_{AB}]$ is the posteriori state of the system A corresponding to the outcome i .

The function $C_B(\omega_{AB})$ is nonnegative, invariant under local unitary transformations and non-increasing under local operations. It coincides with the von Neumann entropy on the set of pure states and with the quantum mutual information on the set of classical-quantum states having form (35) [35, 20, 36].

Proposition 8. *The function $C_B(\omega_{AB})$ is lower semicontinuous on the set $\mathfrak{S}(\mathcal{H}_{AB})$ and*

$$\text{dj}\{C_B(\omega_{AB}^n)\} \doteq \limsup_{n \rightarrow +\infty} C_B(\omega_{AB}^n) - C_B(\omega_{AB}^0) \leq \text{dj}\{H(\omega_A^n)\} \quad (53)$$

for any sequence $\{\omega_{AB}^n\} \subset \mathfrak{S}(\mathcal{H}_{AB})$ converging to a state ω_{AB}^0 .

In particular, local continuity of $H(\omega_A)$ implies local continuity of $C_B(\omega_{AB})$.

Proof. Since for any given measurement $\{M_i\}$ the value in the square bracket in (52) is a lower semicontinuous function of ω_{AB} , the lower semicontinuity of $C_B(\omega_{AB})$ follows from its definition.

To prove (53) we will use the Koashi-Winter relation

$$C_B(\omega_{AB}) + E_F^d(\omega_{AC}) = H(\omega_A) \quad (54)$$

valid for any pure state ω_{ABC} [37], where E_F^d is a discrete version of the entanglement of formation defined by formula (49).¹³

We may assume that $H(\omega_A^n)$ is finite for all n . By Lemma 1 there is a sequence $\{\tilde{\omega}_{ABC}^n\}$ of pure states converging to a state $\tilde{\omega}_{ABC}^0$ such that

¹³A generalizations of the proof of (54) to infinite dimensions is straightforward.

$\tilde{\omega}_{AB}^n = \omega_{AB}^n$ for all $n \geq 0$. Proposition 8 in [31] and the assumed finiteness of $H(\omega_A^n)$ show that

$$\liminf_{n \rightarrow +\infty} E_F^d(\omega_{AC}^n) \geq E_F^d(\omega_{AC}^0).$$

So, Lemma 2 and identity (54) imply (53). \square

The *quantum discord* is the difference between the quantum mutual information and the above measure of classical correlations:

$$D_B(\omega_{AB}) \doteq I(A:B)_\omega - C_B(\omega_{AB}). \quad (55)$$

It is proposed in [38] as quantity describing quantum component of correlations of a state ω_{AB} (see [20, 36, 39] and the references therein).

The quantum discord is not upper or lower semicontinuous. So, its discontinuity for a given sequence $\{\omega_{AB}^n\}$ converging to a state ω_{AB}^0 such that $I(A:B)_{\omega_n} < +\infty$ can be characterised by two nonnegative values

$$\text{dj}^\downarrow\{D_B(\omega_{AB}^n)\} \doteq \max\left\{\limsup_{n \rightarrow +\infty} D_B(\omega_{AB}^n) - D_B(\omega_{AB}^0), 0\right\}$$

and

$$\text{dj}^\uparrow\{D_B(\omega_{AB}^n)\} \doteq \max\left\{D_B(\omega_{AB}^0) - \liminf_{n \rightarrow +\infty} D_B(\omega_{AB}^n), 0\right\}$$

describing respectively the maximal loss and the maximal gain of the quantum discord corresponding to this sequence.

Corollary 10. *Let $\{\omega_{AB}^n\}$ be a sequence converging to a state ω_{AB}^0 such that $I(A:B)_{\omega_n} < +\infty$ for all $n \geq 0$. Then*

$$\begin{aligned} \text{dj}^\downarrow\{D_B(\omega_{AB}^n)\} &\leq \min\{2\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_B^n)\}\} \\ \text{dj}^\uparrow\{D_B(\omega_{AB}^n)\} &\leq \min\{\text{dj}\{H(\omega_A^n)\}, \text{dj}\{H(\omega_{AB}^n)\}\}. \end{aligned} \quad (56)$$

In particular, local continuity of $H(\omega_A)$ implies local continuity of $D_B(\omega_{AB})$.

Proof. All the upper bounds in (56) are proved by applying Lemma 2 to relation (55) and to the following modification of Koashi-Winter relation

$$D_B(\omega_{AB}) + C_B(\omega_{BC}) = H(\omega_B)$$

valid for any pure state ω_{ABC} [36], and by using Theorem 1, Proposition 8, Lemma 1 and the equality $H(\omega_{AB}) = H(\omega_C)$ for a pure state ω_{ABC} . \square

5 Entropic characteristics of quantum channels and operations

5.1 Output entropy of quantum operations

The output entropy $H_\Phi(\rho) \doteq H(\Phi(\rho))$ of a quantum operation $\Phi : A \rightarrow B$ is a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_A)$ of input states. So, its discontinuity for a given sequence $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H}_A)$ converging to a state $\rho_0 \in \mathfrak{S}(\mathcal{H}_A)$ is characterised by the nonnegative value

$$\text{dj}\{H_\Phi(\rho_n)\} \doteq \limsup_{n \rightarrow +\infty} H_\Phi(\rho_n) - H_\Phi(\rho_0),$$

which can be called *the output entropy loss* of the operation Φ corresponding to this sequence (it is assumed that $\text{dj}\{H_\Phi(\rho_n)\} = +\infty$ if $H_\Phi(\rho_0) = +\infty$).

In general, finiteness and local continuity of the von Neumann entropy are not preserved by quantum operations, which means that we can not write general bound for $\text{dj}\{H_\Phi(\rho_n)\}$ in terms of $\text{dj}\{H(\rho_n)\}$. The first part of the following theorem characterizes a class of quantum operations for which such bound exists.

Theorem 2. A) Let $\Phi : A \rightarrow B$ be a quantum operation. The following properties are equivalent:

- (i) there is $C > 0$ such that $\text{dj}\{H_\Phi(\rho_n)\} \leq C \text{dj}\{H(\rho_n)\}$ for any converging sequence $\{\rho_n\}$ of input states;
- (ii) the function H_Φ is continuous and bounded on the set $\text{ext}\mathfrak{S}(\mathcal{H}_A)$;¹⁴
- (iii) the function H_Φ is continuous on the cone $\{\rho \in \mathfrak{T}_+(\mathcal{H}_A) \mid \text{rank} \rho \leq 1\}$.

If these properties hold then $C = 1$ in (i).

B) Let $\Phi : A \rightarrow B$ be a quantum channel and $\hat{\Phi} : A \rightarrow E$ its complementary channel defined by (4). Then for any converging sequence $\{\rho_n\}$ of input states the following inequality holds

$$\text{dj}\{H_\Phi(\rho_n)\} \leq \text{dj}\{H(\rho_n)\} + 2\text{dj}\{H_{\hat{\Phi}}(\rho_n)\},$$

where the factor 2 can be removed if $\{H_{\hat{\Phi}}(\rho_n)\}$ is a converging sequence.

¹⁴ $\text{ext}\mathfrak{S}(\mathcal{H}_A)$ is the set of pure states – extreme points of the set $\mathfrak{S}(\mathcal{H}_A)$.

If $\text{dj}\{H_{\hat{\Phi}}(\rho_n)\}=0$ (in particular, if $\dim E < +\infty$) then

$$\text{dj}\{H_{\Phi}(\rho_n)\}=\text{dj}\{H(\rho_n)\}.$$

Remark 4. By Theorem 2 in [18] properties (ii) and (iii) in Theorem 2A are equivalent to preserving of local continuity of the entropy under action of the operation Φ . So, quantum operations possessing these properties were called PCE-operations in [18]. The simplest examples of PCE-operations are quantum operations with finite Choi rank, for which property (iii) in Theorem 2 is directly verified (since such operations have the Kraus representation with a finite number of summands).

If Φ is a *channel* with finite Choi rank then $\text{dj}\{H_{\Phi}(\rho_n)\}=\text{dj}\{H(\rho_n)\}$ by Theorem 2B.

Proof. A) By Remark 4 it suffices only to show that (ii) implies (i).

According to the general approximating technic used in the proof of Theorem 2 in [18] the functions H and H_{Φ} are pointwise limits of the nondecreasing sequences $\{H_k\}$ and $\{H_{\Phi}^k\}$ of k -order approximators defined for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$ as follows

$$H_k(\rho) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H(\rho_i), \quad H_{\Phi}^k(\rho) = \sup_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H_{\Phi}(\rho_i),$$

where $\mathcal{P}_k(\rho)$ is the set of all countable ensembles with the average state ρ consisting of states of rank $\leq k$. The functions H_k are continuous on $\mathfrak{S}(\mathcal{H}_A)$ for all k by the strong stability of $\mathfrak{S}(\mathcal{H}_A)$ [17] while (ii) implies continuity of all the functions H_{Φ}^k on $\mathfrak{S}(\mathcal{H}_A)$ [18].

By concavity of the function $\eta(x) = -x \log x$ and monotonicity of the relative entropy we have (cf.[18])

$$\begin{aligned} H_{\Phi}(\rho) - H_{\Phi}^k(\rho) &\leq \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\rho)) \\ &\leq \inf_{\{\pi_i, \rho_i\} \in \mathcal{P}_k(\rho)} \sum_i \pi_i H(\rho_i \| \rho) = H(\rho) - H_k(\rho), \end{aligned}$$

for any input state ρ with finite $H(\rho)$. So, the validity of (i) with $C = 1$ follows from Lemma 3 in Section 2.

B) Since

$$H_{\Phi}(\rho) + H_{\hat{\Phi}}(\rho) = H(\rho) + I(B:E)_{V\rho V^*}, \quad \rho \in \mathfrak{S}(\mathcal{H}_A), \quad (57)$$

where $V : A \rightarrow BE$ is any Stinespring isometry for Φ , this assertion follows from Theorem 1 and the lower semicontinuity of all the terms in (57). \square

5.2 Information characteristics of a quantum channel

In this section we will consider three basic characteristics of a quantum channel: the constrained Holevo capacity, the quantum mutual information and the coherent information. We will obtain estimates for discontinuity jumps of these characteristics with respect to simultaneous variations of a channel and of an input state.

The *constrained Holevo capacity* of a quantum channel $\Phi : A \rightarrow B$ at a state $\rho \in \mathfrak{S}(\mathcal{H}_A)$ is defined as follows

$$\bar{C}(\Phi, \rho) = \sup_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\rho)) = H(\Phi(\rho)) - \inf_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i)),$$

where the supremum (the infimum) is over all countable ensembles $\{\pi_i, \rho_i\}$ of input states with the average state ρ and the second formula is valid under the condition $H(\Phi(\rho)) < +\infty$. This quantity plays a basic role in analysis of the classical capacity of a quantum channel (see details in [2, Ch.8]).

The *quantum mutual information* is an important characteristic of a quantum channel related to its entanglement-assisted classical capacity [2, 8]. For a finite-dimensional channel $\Phi : A \rightarrow B$ it can be defined as

$$I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\Phi, \rho),$$

where $H(\Phi, \rho)$ is the entropy exchange of the channel Φ at a state ρ coinciding with the output entropy $H(\hat{\Phi}(\rho))$ of any complementary channel $\hat{\Phi}$ to the channel Φ (see Section 2). In infinite dimensions this definition may contain the uncertainty " $\infty - \infty$ ", but it can be modified to avoid this problem as follows

$$I(\Phi, \rho) = H(\Phi \otimes \text{Id}_R(\hat{\rho}) \| \Phi(\rho) \otimes \varrho), \quad (58)$$

where $\hat{\rho}$ is a purification of the state ρ in $\mathfrak{S}(\mathcal{H}_{AR})$ and $\varrho = \text{Tr}_A \hat{\rho}$. For an arbitrary quantum channel Φ the nonnegative function $\rho \mapsto I(\Phi, \rho)$ defined by (58) is concave and lower semicontinuous on the set $\mathfrak{S}(\mathcal{H}_A)$ [2].

The *coherent information*

$$I_c(\Phi, \rho) \doteq H(\Phi(\rho)) - H(\Phi, \rho) \quad (59)$$

of a channel Φ at a state ρ is an important characteristic related to the quantum capacity of a channel [2, 8]. More suitable representation for the coherent information in infinite dimensions is given by the formula

$$I_c(\Phi, \rho) = I(\Phi, \rho) - H(\rho), \quad (60)$$

where $I(\Phi, \rho)$ is the mutual information defined by (58). This formula correctly determines a value in $[-H(\rho), H(\rho)]$ for any input state ρ with finite entropy (despite possible infinite values of $H(\Phi(\rho))$ and $H(\Phi, \rho) = H(\widehat{\Phi}(\rho))$).

We will obtain estimates for discontinuity jumps of the above characteristic considered as functions of a pair (Φ, ρ) , i.e. as functions on the Cartesian product of the set \mathfrak{F}_{AB} of all quantum channels from A to B equipped with an appropriate topology (type of convergence) and the set $\mathfrak{S}(\mathcal{H}_A)$ of input states. Such consideration is necessary for study of variation of quantum channel capacities with respect to variation of a channel and for analysis of quantum channels by approximation [40]. We will assume that the set \mathfrak{F}_{AB} is equipped with the *strong convergence topology* [40], in which convergence of a sequence $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ to a channel $\Phi_0 \in \mathfrak{F}_{AB}$ means that

$$\lim_{n \rightarrow +\infty} \Phi_n(\rho) = \Phi_0(\rho) \quad \forall \rho \in \mathfrak{S}(\mathcal{H}_A).$$

Preferability of using this topology in the infinite-dimensional case in comparison with the stronger topology induced by the norm of complete boundedness is discussed in Section 8.2. in [5].

The functions $(\Phi, \rho) \mapsto \bar{C}(\Phi, \rho)$ and $(\Phi, \rho) \mapsto I(\Phi, \rho)$ are lower semicontinuous on $\mathfrak{F}_{AB} \times \mathfrak{S}(\mathcal{H}_A)$ [40], so discontinuity jumps of these functions for given sequences $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ and $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H}_A)$ converging respectively to a channel $\Phi_0 \in \mathfrak{F}_{AB}$ and to a state $\rho_0 \in \mathfrak{S}(\mathcal{H}_A)$ are characterised by the nonnegative values

$$\text{dj}\{\bar{C}(\Phi_n, \rho_n)\} \doteq \limsup_{n \rightarrow +\infty} \bar{C}(\Phi_n, \rho_n) - \bar{C}(\Phi_0, \rho_0)$$

and

$$\text{dj}\{I(\Phi_n, \rho_n)\} \doteq \limsup_{n \rightarrow +\infty} I(\Phi_n, \rho_n) - I(\Phi_0, \rho_0)$$

(it is assumed that $\text{dj}\{X(\Phi_n, \rho_n)\} = +\infty$ if $X(\Phi_0, \rho_0) < +\infty$, $X = \bar{C}, I$).

Proposition 9. *For any sequences $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ and $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H}_A)$ converging respectively to a channel $\Phi_0 \in \mathfrak{F}_{AB}$ and to a state $\rho_0 \in \mathfrak{S}(\mathcal{H}_A)$ the following inequalities hold*

$$\text{dj}\{\bar{C}(\Phi_n, \rho_n)\} \leq \text{dj}\{H(\Phi_n(\rho_n))\}, \quad (61)$$

$$\text{dj}\{I(\Phi_n, \rho_n)\} \leq 2 \min \{\text{dj}\{H(\rho_n)\}, \text{dj}\{H(\Phi_n(\rho_n))\}\}. \quad (62)$$

Proof. Let $\overline{\text{co}}H_\Phi$ be the convex closure of the output entropy of the channel Φ – the maximal lower semicontinuous convex function on $\mathfrak{S}(\mathcal{H}_A)$ not exceeding the function $H_\Phi = H(\Phi(\cdot))$. Inequality (61) is proved by applying Lemma 2 to the identity

$$\bar{C}(\Phi, \rho) + \overline{\text{co}}H_\Phi(\rho) = H(\Phi(\rho)) \quad (63)$$

valid for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$, and by using the lower semicontinuity of the function $(\Phi, \rho) \mapsto \overline{\text{co}}H_\Phi(\rho)$ on $\mathfrak{F}_{AB} \times \mathfrak{S}(\mathcal{H}_A)$ [40].

Inequality (62) is proved by applying Theorem 1 and Lemma 1 to representation (58), since for any system R the strong convergence of a sequence $\{\Phi_n\}$ to a channel Φ_0 implies the strong convergence of the sequence $\{\Phi_n \otimes \text{Id}_R\}$ to the channel $\Phi_0 \otimes \text{Id}_R$. We have only to note that $H(\varrho) = H(\rho)$ for the state ϱ in (58). \square

Remark 5. By lower semicontinuity of the function $(\Phi, \rho) \mapsto \bar{C}(\Phi, \rho)$ on $\mathfrak{F}_{AB} \times \mathfrak{S}(\mathcal{H}_A)$ identity (63) and Lemma 2 also imply

$$\text{dj}\{\overline{\text{co}}H_{\Phi_n}(\rho_n)\} \leq \text{dj}\{H(\Phi_n(\rho_n))\}$$

for any sequences $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ and $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H}_A)$ converging respectively to a channel $\Phi_0 \in \mathfrak{F}_{AB}$ and to a state $\rho_0 \in \mathfrak{S}(\mathcal{H}_A)$.

The function $(\Phi, \rho) \mapsto I_c(\Phi, \rho)$ is defined by formula (60) on the set $\mathfrak{F}_{AB} \times \mathfrak{S}_f(\mathcal{H}_A)$, where $\mathfrak{S}_f(\mathcal{H}_A)$ is the set of input states with finite entropy. This function is not upper or lower semicontinuous.¹⁵ So, its discontinuity for given sequences $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ and $\{\rho_n\} \subset \mathfrak{S}_f(\mathcal{H}_A)$ converging respectively to a channel $\Phi_0 \in \mathfrak{F}_{AB}$ and to a state $\rho_0 \in \mathfrak{S}_f(\mathcal{H}_A)$ can be characterised by two nonnegative values

$$\text{dj}^\downarrow\{I_c(\Phi_n, \rho_n)\} \doteq \max\left\{\limsup_{n \rightarrow +\infty} I_c(\Phi_n, \rho_n) - I_c(\Phi_0, \rho_0), 0\right\}$$

and

$$\text{dj}^\uparrow\{I_c(\Phi_n, \rho_n)\} \doteq \max\left\{I_c(\Phi_0, \rho_0) - \liminf_{n \rightarrow +\infty} I_c(\Phi_n, \rho_n), 0\right\}$$

describing respectively the maximal loss and the maximal gain of the coherent information corresponding to these sequences.

¹⁵By using the arguments from the proof of Corollary 5 in Section 3.2 one can show lower semicontinuity of the coherent information $I_c(\Phi, \rho)$ on the set of all pairs (Φ, ρ) , where Φ is a pseudo-diagonal channel and ρ is a state such that $H(\Phi(\rho)) < +\infty$.

Remark 6. The function $(\Phi, \rho) \mapsto H(\Phi, \rho)$ is lower semicontinuous on $\mathfrak{F}_{AB} \times \mathfrak{S}(\mathcal{H}_A)$. This follows from the representation $H(\Phi, \rho) = H(\Phi \otimes \text{Id}_R(\hat{\rho}))$, where $\hat{\rho}$ is a purification of the state ρ in $\mathfrak{S}(\mathcal{H}_{AR})$, and the arguments at the end of the proof of Proposition 9.¹⁶

Corollary 11. *Let $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H}_A)$ be a sequence converging to a state $\rho_0 \in \mathfrak{S}(\mathcal{H}_A)$ such that $H(\rho_n) < +\infty$ for all $n \geq 0$. Then for any sequences $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ converging to a channel $\Phi_0 \in \mathfrak{F}_{AB}$ the following inequalities hold*

$$\begin{aligned} \text{dj}^\downarrow\{I_c(\Phi_n, \rho_n)\} &\leq \min\{2\text{dj}\{H(\rho_n)\}, \text{dj}\{H(\Phi_n(\rho_n))\}\}, \\ \text{dj}^\uparrow\{I_c(\Phi_n, \rho_n)\} &\leq \min\{\text{dj}\{H(\rho_n)\}, \text{dj}\{H(\Phi_n, \rho_n)\}\}. \end{aligned} \quad (64)$$

If $\{H(\rho_n)\}$ is a converging sequence then the factor 2 in the first inequality can be removed.

Proof. The inequalities (64) and $\text{dj}^\downarrow\{I_c(\Phi_n, \rho_n)\} \leq \text{dj}\{H(\Phi_n(\rho_n))\}$ are derived from (59) and (60) by using Lemma 2 and the lower semicontinuity of the functions $(\Phi, \rho) \mapsto H(\Phi(\rho))$, $(\Phi, \rho) \mapsto I(\Phi, \rho)$ and $(\Phi, \rho) \mapsto H(\Phi, \rho)$ (see Remark 6).

If $\text{dj}^\downarrow\{I_c(\Phi_n, \rho_n)\} > 0$ then representation (60) implies

$$\text{dj}^\downarrow\{I_c(\Phi_n, \rho_n)\} \leq \left[\limsup_{n \rightarrow +\infty} I(\Phi_n, \rho_n) - I(\Phi_0, \rho_0) \right] - \left[\liminf_{n \rightarrow +\infty} H(\rho_n) - H(\rho_0) \right].$$

So, the inequality $\text{dj}^\downarrow\{I_c(\Phi_n, \rho_n)\} \leq 2\text{dj}\{H(\rho_n)\}$ follows from Proposition 9 and the lower semicontinuity of the function $H(\rho)$. If $\{H(\rho_n)\}$ is a converging sequence then the second term in the above inequality coincides with $\text{dj}\{H(\rho_n)\}$. \square

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¹⁶We can not use the representation $H(\Phi, \rho) = H(\hat{\Phi}(\rho))$, since in general strong convergence of a sequence $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ to a channel $\Phi_0 \in \mathfrak{F}_{AB}$ does not imply existence of the corresponding sequence $\{\hat{\Phi}_n\}$ of complementary channels strongly converging to a channel $\hat{\Phi}_0$ complementary to the channel Φ_0 .

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